Journal of Mathematical Chemistry, Vol. 42, No. 4, November 2007 (© 2006)
DOI: 10.1007/s10910-006-9152-3

Regular generalized polyomino graphs

Shou-Zhong Wang
Department of Mathematics, Maoming College maoming, Guangdong 525000, P.R. China
E-mail: wangshzh168@163.com

Rong Si Chen*
Center for Discrete Mathematics and Theoretical Computer Science, Fuzhou University, Fuzhou, Fujian 350002, P.R. China

Received 14 February 2006; revised 22 May 2006

A generalized polyomino graph is regular if it can be subjected to a series of special cell-shedding transformations, square by square, down to a single square. In this paper, we give a necessary and sufficient condition to determine whether or not a generalized polyomino graph is regular.

KEY WORDS: square, cell, regular, polyomino, shedding, perfect matching

1. Introduction

Polyomino graphs [1], also called square-cell configurations [2] or chessboards [3] have useful applications in statistical physics and in modeling problems of surface chemistry (cf. Ref. 2 and the references therein). Moreover, many interesting combinatorial subjects can be produced from them, such as hypergraphs [1], domination problem [3], rook polynomials [4], domino polynomials, etc.

A polyomino graph is a connected finite subgraph of the infinite square lattices of the plane such that each interior face is surrounded by a regular square of side length 1 (called a cell) and each edge belongs to at least one square. A generalized polyomino graph G is a graph obtained by deleting some interior vertices and interior edges from a polyomino graph such that there is at least one "hole" (i.e., the interior face which is not a square) and each of its edges belongs to at least one square of G. In this paper, we concentrate ourselves to those generalized polyomino graphs each of which has exactly one "hole".

For convenience, we always place a polyomino graph in question on a plane so that two edges of each square are vertical.

*Corresponding author.

In [2], Harry and Mezey studied the techniques for the reduction of polyomino graphs to simple ones by cell-shedding processes based on an algorithm of symmetrically removing certain peripheral cells from the graph until some reference graphs is obtained. In this paper, we define three simple cell-shedding processes, and give a necessary and sufficient condition to recognize those generalized polyomino graphs that can be subjected to a series of special cell-shedding processes, square by square, down to a single square.

2. Definitions and lemmas

A perfect matching of a graph G is a set of independent edges of G covering all vertices of G. An edge of a graph G is said to be allowed if it belongs to some perfect matching of G and forbidden otherwise. A connected graph G is said to be elementary if all its allowed edges form a connected subgraph of G. It is known that a connected bipartite graph G is elementary if and only if each of its edges is allowed. Let G be a bipartite graph with a perfect matching M and C be a cycle. If the edges of C appear alternately in M and $E(G)\backslash M$ then C is called an M–alternating cycle. The above terminology is due to Lovász and Plummer [5]. For elementary polyomino graphs and elementary generalized polyomino graphs, the following theorems are known:

Theorem 2.1 [6]. Let G be a polyomino graph. Then G is elementary if and only if the perimeter of G is an M– alternating cycle for some perfect matching M of G.

Theorem 2.2 [7]. Let G be a generalized polyomino graph. Then G is elementary if and only if each of the outer and inner perimeters of G is an M– alternating cycle for some perfect matching M of G.

Let G be an elementary plane bipartite graph. A path P of G with odd length is called a reducible ear if all its interior vertices are of degree 2 and $G-P$ is elementary, where $G-P$ is the subgraph of G obtained by deleting the edges and the interior vertices of P from G. A reducible ear decomposition of G is a representation of G in the form $G = x+P_1+P_2+\cdots+P_r$ such that x is an edge, $G_0 = x$, for $1 \leqslant i \leqslant r$, P_i is a path of odd length, $G_i = x + P_1 + P_2 + \cdots + P_i$ and P_i has no other vertices in common with G_{i-1} except its two end vertices. Note that all the $G_i{}'s$ are also elementary.

Zhang and Zhang [8] proved that a plane bipartite graph G is elementary if and only if it is connected and each of its edges is allowed, and that a graph G is elementary and bipartite if and only if G has a reducible ear decomposition. By these results, an elementary plane bipartite graph G is connected and has at least one reducible ear decomposition.

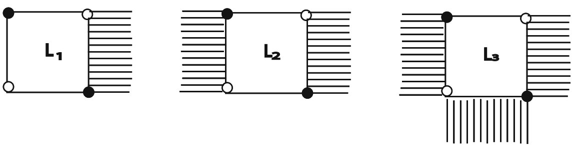

Figure 1. Three modes of a square in a polyomino graph or a generalized polyomino graph.

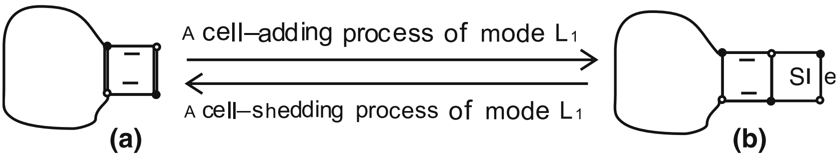

Figure 2. A cell-adding process of mode L_1 (cell-shedding process of mode L_1).

A cell-adding process of mode L_1 (L_3)is adding one square to a polyomino graph or a generalized polyomino graph such that the added square acquires the mode L_1 (L_3) (see figures 1–3).

Since an elementary polyomino graph is a plane bipartite graph, all reducible ear decompositions are in the form $G = x + P_1 + P_2 + \cdots + P_r$, where $x + P_1$ is a square and P_i ($2 \leqslant i \leqslant r$) are paths of either length 3 or 1. Thus an elementary polyomino graph with $h + 1$ squares can be generated from an elementary polyomino graph with h squares by a cell-adding process of mode L_1 or L_3. This fact implies that an elementary polyomino graph can be generated from a single square by a series of cell-adding processes of mode L_1 or L_3, each time only one square is added.

The opposite process of a cell-adding process is a cell-shedding process. Thus, the above fact also implies that an elementary polyomino graph can be subjected to a series of cell-shedding processses of mode L_1 or L_3, square by square, right down to a single square.

Note that after a special cell-adding process of mode L_2, i.e., adding one square to a polyomino graph such that the added square acquires the mode L_2, then a hole appears. This means that the graph obtained by a cell-adding process of mode L_2 is a generalized polyomino graph. It is obvious that a polyomino

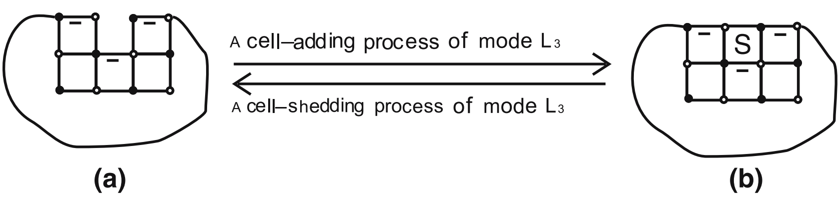

Figure 3. A cell-adding process of mode L_3 (cell-shedding process of mode L_3).

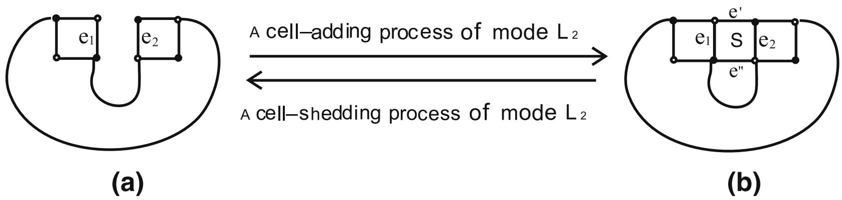

Figure 4. A cell-adding process of mode L_2 (cell-shedding process of mode L_2).

graph is obtained from a generalized polyomino graph by a cell-shedding process of mode L_2 from a generalized polyomino graph (see figure 4).

Definition 2.3. A generalized polyomino graph is regular if it can be subjected to a series of cell-shedding processes of mode L_1 or mode L_3 plus one cell-shedding process of mode L_2, square by square, down to a single square.

Evidently, any regular generalized polyomino graph can be generated from a single square by a series of cell-adding processes of mode L_1 or mode L_3 plus one cell-adding process of mode L_2.

We define the outer perimeter C_0 of a generalized polyomino graph G to be the perimeter of the external region of G, the inner perimeter C_i of G to be the perimeter of the unique hole of G. A perimeter of G is either the outer perimeter C_0, or the inner perimeter C_i. A vertex not belonging to the perimeters of G is said to be an interior vertex of G.

A square of a generalized polyomino graph G is said to be a perimeter square of G if it has at least one edge lying on the perimeter of G; otherwise, it is said to be an inner square of G. An edge of G is said to be perimeter edge if it lies on the perimeter of G; otherwise, it is said to be an inner edge of G. An inner edge of G is said to be a chord if its two end-vertices are on the outer perimeter of G. Let G be a generalized polyomino graph with a chord (denoted by e). It is not difficult to see that G is separated by e into two parts (denoted by G_e and $G_e^{'}$, respectively), one of them is a generalized polyomino graph , the other is a polyomino graph such that each of them has a copy of e. For convenience, we assume G_e that is a generalized polyomino graph and $G_e^{'}$ is a polyomino graph (see figure 5).

Lemma 2.4. Let G be a generalized polyomino graph with a perfect matching M such that both the inner and the outer perimeters of G are M-alternating cycles, and e be a chord of G. Then G_e is a generalized polyomino graph such that both the inner and the outer perimeters of G_e are M'-alternating cycles for some perfect matching M' of G_e, and $G_e^{'}$ is an elementary polyomino graph.

Proof. Let G be a generalized polyomino graph with a perfect matching M such that both the inner and the outer perimeters of G are M- alternating cycles,

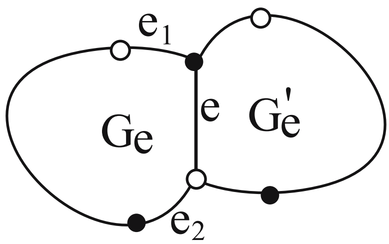

Figure 5. e is a chord of G.

and e be a chord of G. Then the two edges of M covering the end vertices of e belong to one of G_e and G'_e simultaneously because the number of vertices of G_e and G'_e are both even. Without loss of generality, let $e_1, e_2 \in M$ (see figure 5), then $M' = M \cap E(G_e)$ is a perfect matching of G_e such that both outer and inner perimeters of are M'-alternating cycles. $M^* = (M \cup \{e\} \cap E(G'_e))$ is a perfect matching of G'_e such that the perimeter of G'_e is an M^*-alternating cycle, by theorem 2.1, G'_e is an elementary polyomino graph.

3. Regular generalized polyomino graphs

We are now in the position to formulate our main result.

Theorem 3.1. A generalized polyomino graph G is regular if and only if there is a perfect matching M of G such that both the inner and the outer perimeters of G are M-alternating cycles.

Proof of necessity. Suppose that G is a regular generalized polyomino graph. By definition, G can be generated from a single square by a series of cell-adding processes of mode L_1 or L_3 plus one cell-adding process of mode L_2 in three steps:
Step 1. A polyomino graph G_1 is obtained by a series of cell-adding processes of mode L_1 or L_3 to a single square. Each time only one square is added.
Step 2. A generalized polyomino graph G_2 is formed by one cell-adding process of mode L_2 to G_1.
Step 3. The final generalized polyomino graph G is obtained by a series of cell-adding processes of mode L_1 or L_3 to G_2.

By the theory of reducible ear decomposition for elementary bipartite graphs, G_1 is elementary since G_1 is obtained by a series of cell-adding processes of mode L_1 or L_3 to a single square. Hence the perimeter of G_1 is an M_1-alternating cycle for some perfect matching M_1 of G.

Suppose that square S is added in step 2, i.e. S is a square of mode L_2 (cf. figure 4). Because both a square and a generalized polyomino are plane bipartite

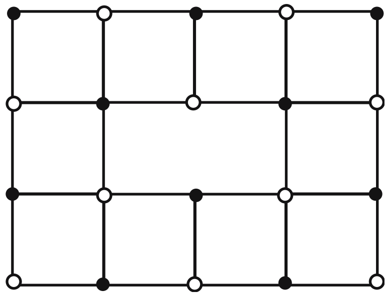

Figure 6. The smallest regular generalized polyomino graph.

graphs, they have no cycle with odd length. Then either e_1 and e_2 belong to M_1 simultaneously; or e_1 and e_2 do not belong to M_1 simultaneously.

If e_1 and e_2 do not belong to M_1, let $M_2 = M_1$, then M_2 is a perfect matching of G_2, and both the outer perimeter and inner perimeter of G_2 are M_2-alternating cycles.

If e_1 and e_2 belong to M_1. Let $E(C)$ be the edge set of the perimeter of G_1, let $M_2 = M_1 \bigoplus E(C)$, where $M_2 = M_1 \bigoplus E(C)$ denotes the symmetric difference of M_1 and $E(C)$. Then M_2 is a perfect matching of G_2 such that both the outer and inner perimeters of G_2 are M_2-alternating cycles.

In step 3, G is obtained by a series of cell-adding processes of mode L_1 or L_3 to G_2. It is not difficult to check that both the inner and the outer perimeters of G are M-alternating cycles for some perfect matching M of G. The proof of necessity is thus completed.

Proof of sufficiency. Suppose that a generalized polyomino graph G has a perfect matching M such that both the inner and the outer perimeters of G are M-alternating cycles. We want to prove that G is subjected to a series of cell-shedding processes of mode L_1 or mode L_3 plus one cell-shedding process of mode L_2, square by square, down to a single square. We proceed by induction on the number of squares of G. It is not difficult to see that the smallest generalized polyomino graph satisfing the condition of the sufficiency is the one with 10 squares as depicted in figure 6. One can check that it is regular.

Now suppose that G has h $(h > 10)$ squares. We distinguish three cases.

Case 3.1. G has a square of mode L_2. If G has a perfect matching M such that both the inner and outer perimeters of G are M-alternating cycles, then G has four perfect matchings with the same property. Without loss of generality, assume that M is one of these perfect matchings such that e' and e'' do not belong to M (see figure 4(b)). G' is obtained by one cell-shedding process of mode L_2. Let $M' = M$, it is evident that M' is a perfect matching of G' and the

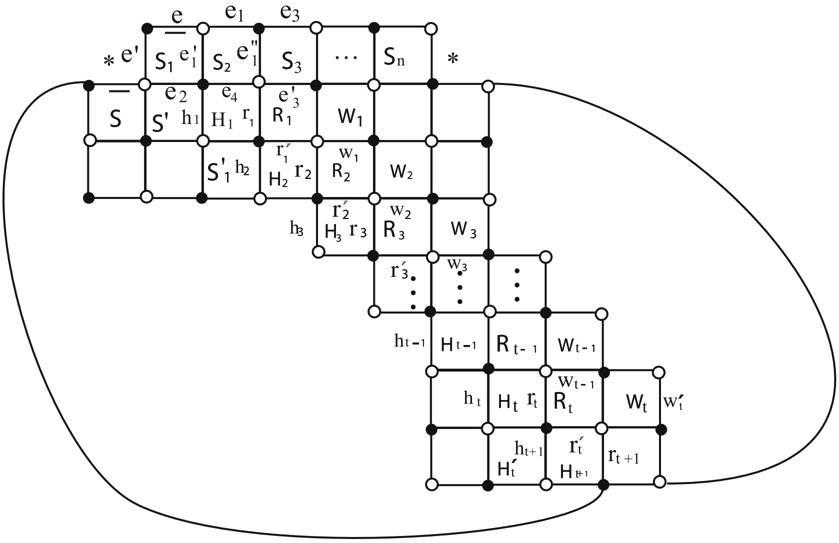

Figure 7. Illustrations for subcase 3.1.

perimeter of G' is a M'- alternating cycle. By induction hypothesis G' is regular. Therefore, G is regular.

Case 3.2. G has a square of mode L_1. It is obvious that G has a perfect matching M_1 such that both the outer and inner perimeters of G are M_1-alternating cycles and $e \in M_1$ (see figure 2(b)). G' is obtained by a cell-shedding process of mode L_1. Let $M' = M - \{e\}$, then M' is a perfect matching of G' such that both outer and inner perimeters of G' are M'-alternating cycles (see figure 2(a)). By induction hypothesis, G' is regular. So is G.

Case 3.3. G has no square of mode L_1 or L_2. G can be placed on the plane in four different positions such that in each position two edges of each square are parallel to the vertical line. For each of the four possible positions of G, for each row we can label a series of outer perimeter squares $S_1, S_2, \ldots, S_n$ as shown in figure 7. The squares S_{i-1} and S_i $(2 \leqslant i \leqslant n)$ have one edge in common. While there is no square belonging to G on the left-hand side of S_1 and on the right-hand side of S_n (cf. figure 7, marked by star).

Subcase 3.1. G has a top row with $n \geqslant 3$ for some of its four possible positions. First, we consider the case that square H_1 belongs to G. We claim that G contains square S'. In fact, if G does not contain square S, then G must contain square S' since G has no square of mode L_1. If G contains square S, suppose G does not contain square S', by the necessary condition of the theorem, G

has a perfect matching M such that the inner and outer perimeters of G are M-alternating cycles. By theorem 2.2, G is elementary. This fact implies G is 2-connected. So the common vertex v of square S and S' is not a cut-vertex, i.e. v lies on both the inner and the outer perimeters of G. Then v is covered by two edges belonging to M. One of them is an edge of the inner perimeter, the other is an edge of the outer perimeter of G. Contradicting that M is a perfect matching of G. Thus G must also contain square S' (see figure 7). Without loss of generality, let $e, e_3 \in M$.

Subcase 3.1.1. If $e_4 \in M$, then let $G' = G - \{e, e', e_1, e'_1\}$, i.e. G' is obtained by a cell-shedding process of mode L_3 (cf. figure 7, square S_2) and a cell-shedding process of mode L_1 (cf. figure 7, square S_1) . Let $M' = M - \{e\}$, then M' is a perfect matching of G' such that both the outer and the inner perimeters of $G'$are M' -alternating cycles.

Subcase 3.1.2. If $e_4 \bar{\in} M$, then $h_1 \in M$. Note that G contains squares S' and H_1, then h_1 does not lie on the perimeters of G. So G contains square S'_1.

Subcase 3.1.2.1. If $r_1 \in M$. Then the perimeter of H_1 is an M-alternating cycle. We claim r_1 is not an outer perimeter edge. Otherwise, let $E(H_1)$ be the edge set of the perimeter of H_1, let $M^* = M \bigoplus E(H_1)$, then $e \in M^*$ and $e_4 \in M^*$. On the other hand, let $G' = G - \{r_1\}$, then G' is a generalized polyomino graph, and both the inner and the outer perimeters of G' are M^*-alternating cycles. Because e'_1 is a chord of G'. By the proof of lemma 2.4, the two edges of M^* covering the end vertices of e'_1 either belong to $G_{e'_1}$ simultaneously or belong to $G'_{e'_1}$ simultaneously. It contradicts that $e \in M^*$ and $e_4 \in M^*$. Since G has no square of mode L_2, then r_1 is not an inner perimeter edge. So r_1 is an inner edge, i.e. H_1 is an inner square of G. Let $M^* = M \bigoplus E(H_1)$, then $e_4 \in M^*$, i.e. the case is similar to subcase 3.1.1. G' is obtained by cell-shedding process of mode L_3 and cell-shedding process of mode L_1. So both the inner and the outer perimeters of G' are M^*-alternating cycles.

Subcase 3.1.2.2. If $r_1 \bar{\in} M$, then $e'_3 \in M$. Assume r_1 is an outer perimeter edge of G, then let $G' = G - \{r_1\}$, i.e. G' is obtained by cell-shedding process of mode L_3 . Let $M' = M$, then M' is a perfect matching of G' such that both the outer and the inner perimeters of G' are M'-alternating cycles.

Assume $r'_1 \in M$ and R_1 is an inner square. Let H^* be the subgraph of G which contains only squares S_4, R_1, and $C(H^*)$ be the set of edges of perimeter of H^*. Then H^* also is a polyomino graph and $C(H^*)$ is an M-alternating cycle. Let $M^* = M \bigoplus C(H^*)$, then $e, e_4 \in M^*$. It can be dealt with in a similar way as in subcase 3.1.1., we omit the details.

Assume $r_1' \in M$ and r_1' is an inner perimeter edge. By the above discussion, G must contain square W_1. Let $E(C'')$ be the edge set of the inner perimeter of G, $M' = M \bigoplus E(C'')$ and $G' = G - \{r_1', r_1, e_4, e_1\}$, G' is obtained by cell-shedding process of mode L_3, and one cell-shedding process of mode L_2. Then M' is a perfect matching of G' such that the perimeter of G' is M'-alternating cycle. So G' is an elementary polyomino graph.

Repeating the above discussion, without loss of generality, we assume that $h_1, h_2, \ldots, h_t, w_1, w_2, \ldots, w_{t-1} \in M$; $S', H_1, H_2, \ldots, H_t, R_1, \ldots, R_{t-1}$ are inner squares of G and W_t is an outer perimeter square of G.

Since W_t is a square of outer perimeter of G, w_t' is an outer perimeter edge of G, M covers two end vertices of w_t'. Then there must exist the following three cases: (a) $r_t' \in M$ and r_t' is a perimeter edge of G; (b) $r_t' \in M$ and R_t is an inner square of G; (c) $h_{t+1}, r_{t+1} \in M$.

Subcase 3.1.2.2.1. If $r_t' \in M$, we claim r_t' is not an outer perimeter edge. Otherwise, let H^* be a subgraph of G which contains exactly squares $H_1, R_1, H_2, R_2, \ldots, H_t, R_t$, and $C(H^*)$ be the set of edges of the perimeter of H^*. Then H^* is a polyomino graph and $C(H^*)$ is an M-alternating cycle. Let $M^* = M \bigoplus C(H^*)$, then $e_4 \in M^*$ and $e \in M^*$. On the other hand, let $G' = G - \{r_t', r_t, \ldots, r_1', r_1\}$, then both the inner and the outer perimeters of G' are M^*-alternating cycles. Because e_1' is a chord of G', by the proof of lemma 2.4, the two edges of M^* covering the end vertices of e_1' either belong to $G_{e_1'}$ or belong to $G'_{e_1'}$, simultaneously. It contradicts with $e \in M^*$ and $e_4 \in M^*$. Then we have the following two cases:

(1) Assume R_t is an inner square of G. H^* and $C(H^*)$ are defined as above. Let $M^* = M \bigoplus C(H^*)$, then $e_4 \in M^*$ and M^* is a perfect matching of G such that both the outer and the inner perimeters of G are M^*-alternating cycles. It can be dealt with in a similar way as in subcase 3.1.1., we omit the details.

(2) Assume r_t' is an inner perimeter edge of G, let $M' = M \bigoplus E(C'')$ and $G' = G - \{r_t', r_t, \ldots, r_1', r_1, e_4, e_1\}$, i.e. G' is obtained by cell-shedding processes of mode L_3, and a cell-shedding process of mode L_2 from G. Hence G' is an elementary polyomino graph.

Subcase 3.1.2.2.2. If $r_t' \bar{\in} M$. Similarly to the above discussion, we know that neither r_{t+1} is an outer perimeter edge of G, nor r_{t+1} is an inner perimeter edge of G. So, we have the following three cases:

(1) Assume r_t' is an outer perimeter edge of G, then let $G' = G - \{r_t', r_t, \ldots, r_1', r_1\}$, i.e. G' is obtained by cell-shedding processes of mode L_3 from G. Let $M' = M$, then M' is a perfect matching

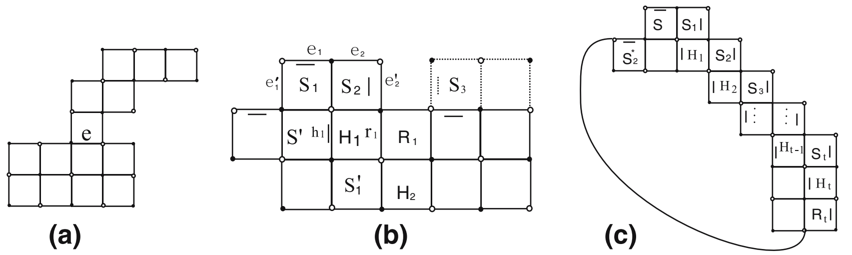

Figure 8. (a) Illustration for subcase 3.2. (b) Illustration for subcase 3.3.1. (c) Illustration for subcase 3.3.2.

of G' such that both the outer and the inner perimeters of G' are M'-alternating cycles. Hence G' is a generalized polyomino graph.

(2) Assume r_t' is an inner perimeter edge of G, let $M' = M$ and $G' = G - \{r_t', r_t, \ldots, r_1', r_1, e_4, e_1\}$, i.e. G' is obtained by cell-shedding processes of mode L_3 from G , and one cell-shedding process of mode L_2 from G. Hence G' is an elementary polyomino graph.

(3) Assume $h_{t+1}, r_{t+1} \in M$, and H_{t+1} is an inner square of G, it can be dealt with in a similar way as in subcase 3.1.2.2.1 (1), we omit the details.

Subcase 3.2. Now consider the case that H_1 does not belong to G. Since G contains no square of mode L_1, there exists a square S' which is of mode L_3 (see figure 8(a)). Otherwise, it contradicts the reducible ear decompositions of an elementary plane bipartite graph [8]. Without loss of generality, we may assume G_e contains squares of mode L_3. Then G_e can be reduced to subcase 3.1 or 3.3.

Subcase 3.3. For each of the four possible positions, for each top row of G, we always have $n = 2$. Similar to the discussion of subcase 3.1, we know that G contains S'. For M, without loss of generality, let $e_1, e_2' \in M$. Then $h_1 \in M$. Thus h_1 is an inner edge of G. Hence G contains square S_1'.

If G does not contain the square R_1 (G may contain the square S_3, see figure 8(b)). Then H_1 is outer perimeter square of G. Let G' be obtained by elementary tearing down H_1 in mode L_3 from G and elementary tearings down $S_1$and S in mode L_1 from G. Let $M' = M - \{e_1, e_2'\}$, then M' is a perfect matching of G' such that both the outer and the inner perimeters of G' are M'-alternating cycles.

Subcase 3.3.1. There is at least one square in the right side of S_2 (see figure 8(b)), then it can be dealt in a similar way as in subcase 3.1, we omit the details.

Subcase 3.3.2. There is no square on the right side of S_2. Assume there are squares on the right side of $S_i (3 \leqslant i \leqslant n-1)$, then the discussion is entirely similar to the discussion of subcase 3.3.1. Without loss of generality, suppose that there is no square in the right side of $S_2, S_3, \ldots, S_t$ and S_t is a square in the outer perimeter of G (see figure 8(c)). Since G has no square of mode L_1, G must contain H_t and R_t. Otherwise, G will have no perfect matching. Thus we can execute a series of cell-shedding processes of mode L_3 and L_1 from G, and obtain graph G' such that both the inner and the outer perimeters of G' are alternating cycles of the same perfect matching.

By induction hypothesis, G' can be subjected to a series of cell-shedding process of modes L_1 and L_3 plus one cell-shedding process of mode L_2. Then G can be subjected to a series of cell-shedding process of modes L_1 and L_3 plus one cell-shedding process of mode L_2. So G is regular. The proof is completed.

Acknowledgment

This work is supported by NSFC (10431020).

References

[1] C. Berge, C.C. Chen, V. Chvatal and C.S. Seow, Combinatorial properties of polyominoes, Combinatorial 1(3) (1981) 217–224.
[2] F. Harary and P.G. Mezey, Cell-shedding transformations, equivalence relations, and similarity measures for square-cell configurations, Int. Quant. Chem. 62 (1997) 353–361.
[3] E.J. Cockayne, Chessboard domination problems, Discrete Math. 86 (1990) 21–26.
[4] A. Motoyama and H. Hosoya, King and domino polynomials for polyomino graphs, J. Math. Phys. 18 (1977) 1485–1490.
[5] L. Lovász and M.D. Plummer, *Matching Theory*, Annals of discrete mathematics 29 (North-Holland, Amsterdam, 1986).
[6] H.P. Zhang, The connectivity of Z-transformation graphs of perfect matchings of polyominoes. Discrete Math. 158 (1996) 257–272.
[7] R.S. Chen, Perfect matchings of generalized polyomino graphs, Graph. Comb. 21 (2005) 515–529.
[8] F.J. Zhang and H.P. Zhang, Plane elementary bipartite graphs, Discrete Appl. Math. 105 (2000) 291–311.